Carleton University

Investigation of Kronecker-based Recovery in Compressive Sensing [1]

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Supervisor: Prof. Sreeraman Rajan

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[1] Author acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC) & Carleton University.

Compressive Sensing (CS)

Original signal, $\mathbf{x}_{N\times 1}$

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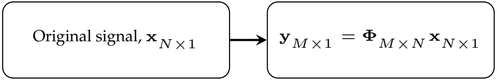

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$\mathbf{y}_{M\times 1} = \mathbf{\Phi}_{M\times N}\mathbf{x}_{N\times 1}$

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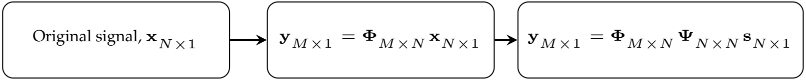

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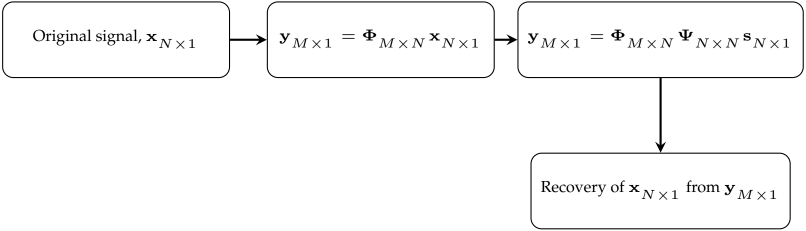

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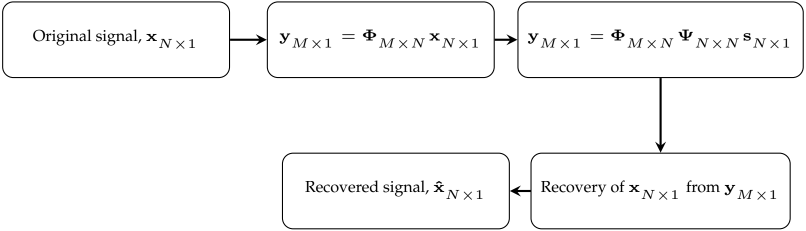

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Recovery of $\mathbf{x}_{N \times 1}$ from $\mathbf{y}_{M \times 1}$

Recovered signal, $\hat{\mathbf{x}}_{N \times 1}$

Measurement Matrix Construction

$\mathbf{\Phi}_{M\times N}$ for a k-sparse signal $\mathbf{x}_{N\times 1}$ should maintain Restricted Isometry Property(RIP) [2] property for successful recovery.

$$(1-\delta_k) \leq \frac{\|\mathbf{\Phi x}\|_2^2}{\|\mathbf{s}\|_2^2} \leq (1+\delta_k) \tag{1}$$

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Does an alternative exist?

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Measurement Matrix Construction (Continued)

- $\mathbf{x}$ can not have sparse representation in both $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ [3].

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- Mutual coherence (μ) between $\mathbf{\Phi}$ and $\mathbf{\Psi}$ is chosen to be an alternative to RIP [4].

$$\mu(\mathbf{\Phi}, \mathbf{\Psi}) = \max_{i \neq j} | < \Phi_i, \Psi_j > | \qquad (2)$$

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- Lower the μ, better the reconstruction.

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Measurement Matrix Construction (Continued)

Selection of $\mathbf{\Phi}$

Random Matrix → Gaussian or Normal, Bernoulli etc.
Deterministic matrix → DBBD, Toeplitz-structured matrix, second-order Reed Muller code based matrix etc.

Measurement Matrix Construction (Continued)

Selection of $\mathbf{\Phi}$

Random matrices →

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- Difficult to implement in hardware.

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Deterministic matrices →

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- Reconstruction quality improves compared to the random matrices for a fixed $\mathbf{\Psi}$.

CS-based Recovery

How to find the sparse solution to the linear system?

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- ✓ $\min||\mathbf{s}||_0$ subject to $\mathbf{y} = \mathbf{\Phi\Psi s}$ → Non-convex optimization problem, NP hard to solve.

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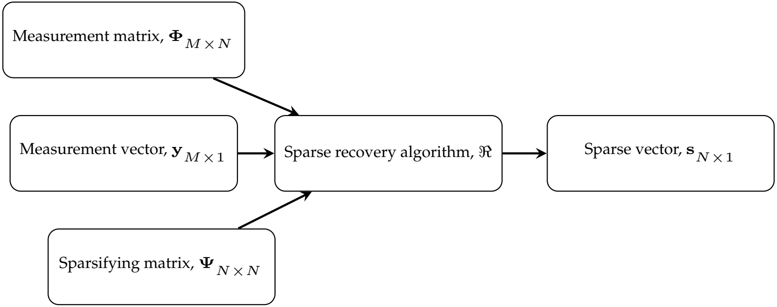

Measurement matrix, $\mathbf{\Phi}_{M \times N}$

Measurement vector, $\mathbf{y}_{M \times 1}$

Sparsifying matrix, $\mathbf{\Psi}_{N \times N}$

Sparse recovery algorithm, $\Re$

Sparse vector, $\mathbf{s}_{N \times 1}$

Figure: Block diagram representation of sparse reconstruction

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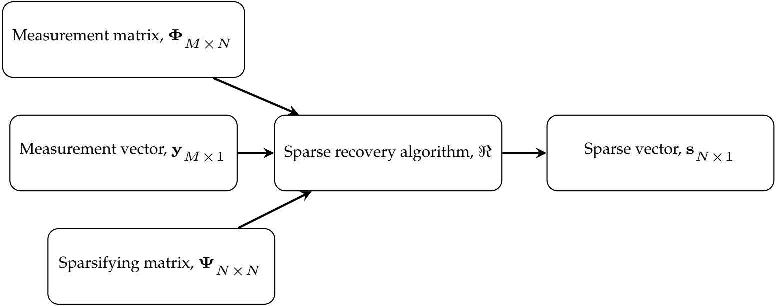

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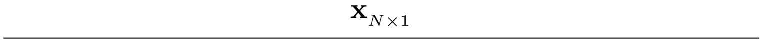
Segmentation-based Sensing



$\mathbf{x}_{N\times 1}$

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$$\overbrace{\underbrace{\qquad\qquad\qquad}_{\mathbf{x}_{N'\times 1}}\underbrace{\qquad\qquad\qquad}_{\mathbf{x}_{N'\times 1}}\underset{p\text{ smaller segments}}{\qquad\qquad\qquad\qquad\qquad\qquad}\underbrace{\qquad\qquad\qquad}_{\mathbf{x}_{N'\times 1}}}^{\mathbf{x}_{N\times 1}\ \text{(where } N = N' \times p\text{)}}$$

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$$\Big\downarrow \boldsymbol{\Phi}_{M'\times N'}$$

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Discussions

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Applications:

1. Continuous bio-signal (e.g. ECG) monitoring using 'resource constraint' wearable devices.

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Applications:

1. Continuous bio-signal (e.g. ECG) monitoring using 'resource constraint' wearable devices.
2. Smaller $\mathbf{\Phi}$ enables segmented column/row-based sensing of larger images.

Segmentation-based Recovery

$$\overbrace{\underbrace{\qquad\qquad}_{\mathbf{y}_{M'\times 1}}\underbrace{\qquad\qquad}_{\mathbf{y}_{M'\times 1}}\underset{p \text{ smaller segments}}{\qquad\qquad\qquad\qquad\qquad\qquad}\underbrace{\qquad\qquad}_{\mathbf{y}_{M'\times 1}}}^{\mathbf{y}_{M\times 1} \text{ (where } M = M' \times p\text{)}}$$

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Challenges

1. Performing recovery p times → Computationally expensive.

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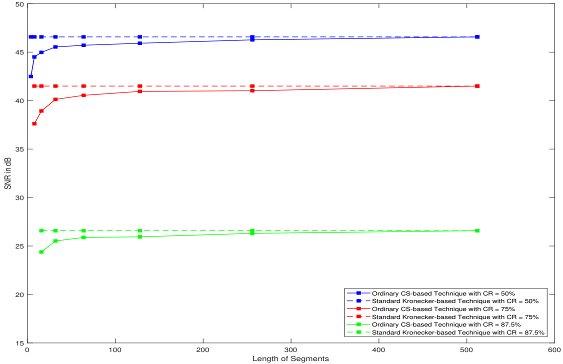

Figure: Reconstructed signal quality degradation caused by segmentation.

Way out?

Question?

Concatenate p individual segments of $\mathbf{y}$ → Form $\mathbf{y}_{M\times1}$ → Perform recovery once.

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$\mathbf{\Phi}_{M'\times N'}$ constructed during measurement → $\mathbf{\Phi}_{M\times N}$ can't be reconstructed!

Way out?

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Concatenate p individual segments of $\mathbf{y} \rightarrow$ Form $\mathbf{y}_{M\times 1} \rightarrow$ Perform recovery once.

Issue

$\mathbf{\Phi}_{M'\times N'}$ constructed during measurement $\rightarrow \mathbf{\Phi}_{M\times N}$ can't be reconstructed!

Solution?

$\mathbf{\Phi}_{M'\times N'}$ can be expanded to form $\mathbf{\Phi}_{M\times N}$.

Kronecker Product-based $\boldsymbol{\Phi}$ Expansion

$$\hat{\boldsymbol{\Phi}}_{M \times N}$$

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$$\hat{\mathbf{\Phi}}_{M\times N} = \mathbf{I}_{p\times p} \otimes \mathbf{\Phi}_{M'\times N'} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi} & \ldots & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{\Phi} \end{bmatrix}$$

Standard Kronecker-based Recovery

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$$\mathbf{s}_{N\times 1} = \Re(\mathbf{y}_{M\times 1}, \hat{\mathbf{\Phi}}_{M\times N}, \hat{\mathbf{\Psi}}_{N\times N})$$

Discussion

Advantage

Computationally expensive recovery algorithm performs once, not p times.

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Quality degradation not addressed.

Modified Kronecker-based Recovery

$$\mathbf{\Psi'}_{N\times N} = \left[\begin{array}{cccc} \mathbf{\Psi'}_{1,1} & \mathbf{\Psi'}_{1,2} & \cdots & \mathbf{\Psi'}_{1,p} \\ \mathbf{\Psi'}_{2,1} & \mathbf{\Psi'}_{2,2} & \cdots & \mathbf{\Psi'}_{2,p} \\ & & \ddots & \\ \mathbf{\Psi'}_{p,1} & \mathbf{\Psi'}_{p,2} & \cdots & \mathbf{\Psi'}_{p,p} \end{array}\right]$$

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$$\mathbf{s}_{N\times 1} = \Re(\mathbf{y}_{M\times 1}, \underset{\uparrow}{\hat{\mathbf{\Phi}}_{M\times N}}, \underset{\uparrow}{\mathbf{\Psi'}_{N\times N}})$$

Remains unchanged

Regenerated from the same basis

Discussion

Theorem

Let us consider the resultant Kronecker-based sparsifying matrix $\hat{\mathbf{\Psi}}_{N \times N} = \mathbf{I}_{p \times p} \otimes \mathbf{\Psi}_{N' \times N'}$ is of size $N \times N$. If the modified sparsifying basis $\mathbf{\Psi}'_{N \times N}$ is regenerated from the same basis then,

$$\mu(\hat{\mathbf{\Phi}}_{M \times N}, \hat{\mathbf{\Psi}}_{N \times N}) \geqslant \mu(\hat{\mathbf{\Phi}}_{M \times N}, \mathbf{\Psi}'_{N \times N})$$

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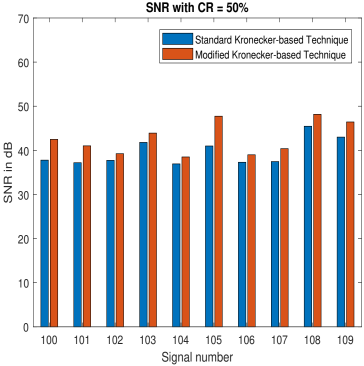
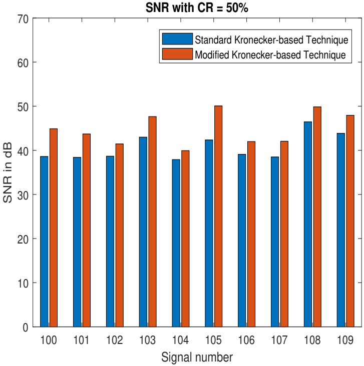
Kronecker CS-based 1-D Signal Recovery



(a) Reconstructed signal[5] with random matrix (Normal distribution) at CR = 50%



(b) Reconstructed signal with random matrix (Bernoulli distribution) at CR = 50%

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[5] "MIT-BIH Arrhythmia Database." Available: `https://www.physionet.org/physiobank/database/mitdb/`

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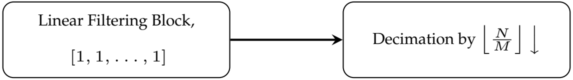

Linear Filtering Block,

$[1, 1, \ldots, 1]$

Decimation by $\left\lfloor \frac{N}{M} \right\rfloor \downarrow$

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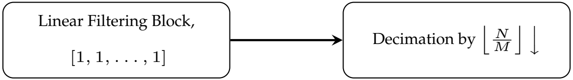

Linear Filtering Block,

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Decimation by $\left\lfloor \frac{N}{M} \right\rfloor \downarrow$

- A matrix representation of DBBD deterministic matrix for $M = 4$ and $N = 16$.

$$\boldsymbol{\Phi}_{4\times 16} = \begin{bmatrix} [1111] & & & \\ & [1111] & & \\ & & [1111] & \\ & & & [1111] \end{bmatrix}$$

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✓ Enables signal processing in the compressed domain.

Kronecker CS-based 1-D Signal Recovery

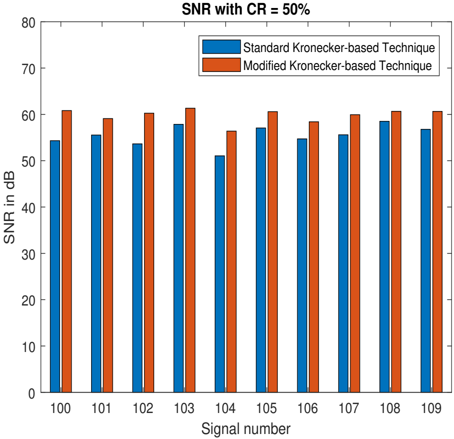

Figure: Reconstructed signal with DBBD deterministic matrix at CR $= 50\%$

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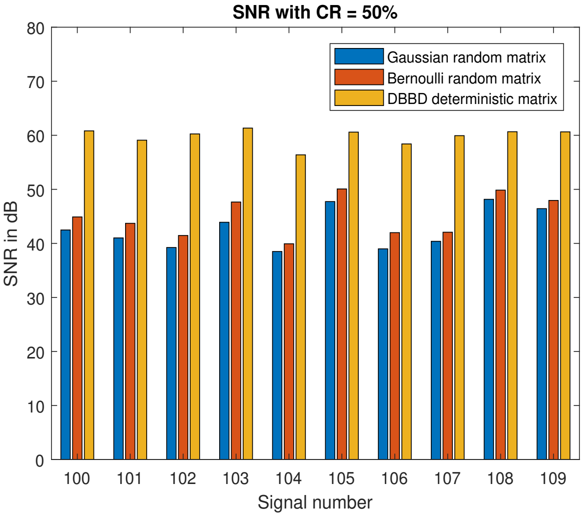


Figure: Comparison of signal quality using random and deterministic matrices at CR $= 50\%$

Kronecker CS-based 1-D Signal Recovery

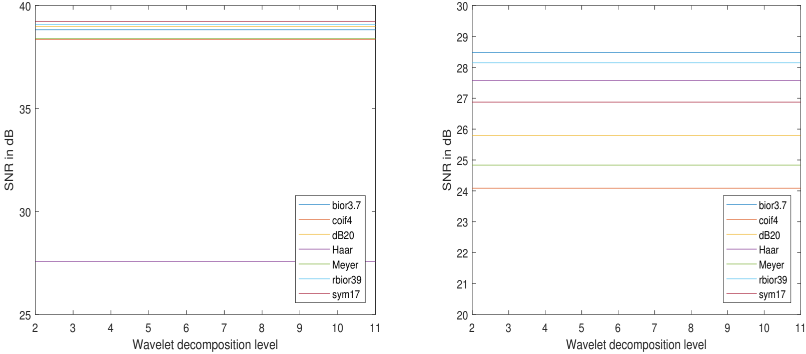
Statistical Parameter	SNR (dB)							
	Biorthogonal	Coiflets	Daubechies	DCT	Haar	Discrete Meyer	Reverse Biorthogonal	Symlets
Minimum	5.31	21.69	17.45	35.17	20.79	31.19	12.46	17.45
Maximum	26.61	23.65	26.76	35.17	20.79	31.19	27.00	36.62
Median	21.96	22.90	23.20	35.17	20.79	31.19	23.08	21.74

Table: Statistical analysis of recovery performance using modified Kronecker-based technique: CR $= 50\%$, $\boldsymbol{\Phi}$ = DBBD

$\boldsymbol{\Phi}$	$\boldsymbol{\Psi}$	Standard				Modified			
		SNR	RMS	QS	MAX	SNR	RMS	QS	MAX
DBBD	DCT	34.38	1.02	1.4	5.85	35.17	1.12	1.48	4.03
Bernoulli Random Matrix	DCT	17.56	7.04	0.18	55.67	20.07	9.87	0.24	30.63
Gaussian Random Matrix	DCT	17.04	7.68	0.16	57.26	19.37	10.62	0.23	36.82

Table: Performance analysis for various measurement matrices at CR $= 50\%$

Kronecker CS-based 1-D Signal Recovery



(a) (b)

Figure: Comparison between different decomposition levels of different wavelets at CR = 50% with $\mathbf{\Phi}$ =DBBD: (a) Using modified Kronecker-based technique, (b) Using standard Kronecker-based technique.

Kronecker CS-based 1-D Signal Recovery

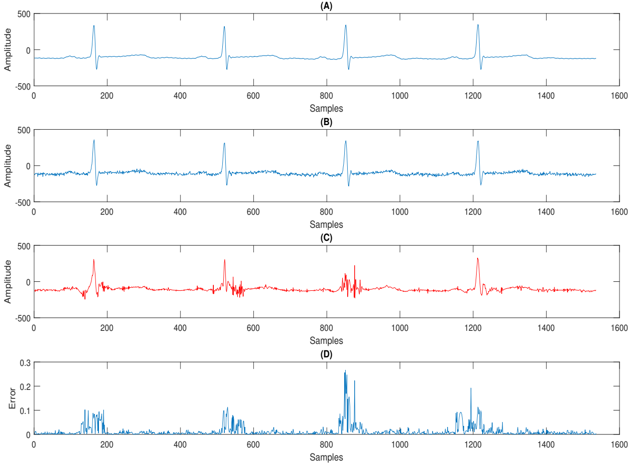


Figure: Visual representation of ECG signals in presence of -15 dB correlated noise using $\mathbf{\Phi}$ = random Bernoulli matrix & $\mathbf{\Psi}$ = Coif5 at CR = 75%.

Kronecker CS-based 1-D Signal Recovery

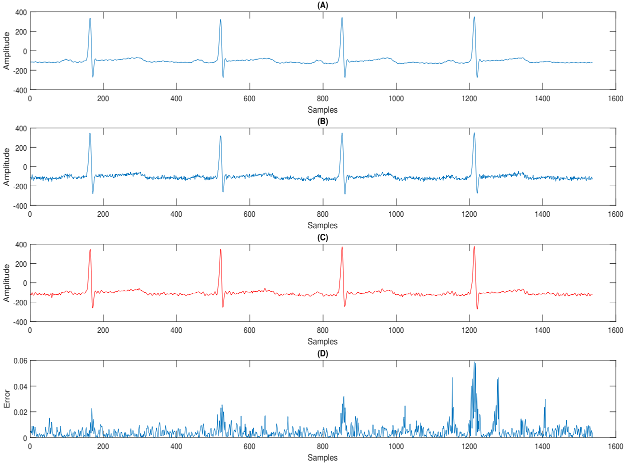


Figure: Visual representation of ECG signals in presence of -15 dB correlated noise using $\mathbf{\Phi}$ = DBBD & $\mathbf{\Psi}$ = Coif5 at CR = 75%.

Discussions

✓ Improvement observed for various levels of compression (50%, 75%, 87.5%).

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✓ Better signal quality is attained by using deterministic measurement matrix.

✓ Wavelet decomposition level does not have impact on signal quality.

✓ DCT serves as better $\boldsymbol{\Psi}$ for DBBD measurement matrix.

✓ Filtering effect of DBBD makes it suitable for measuring noisy signals.

Extension to 2-D Signals

Column-wise Measurement

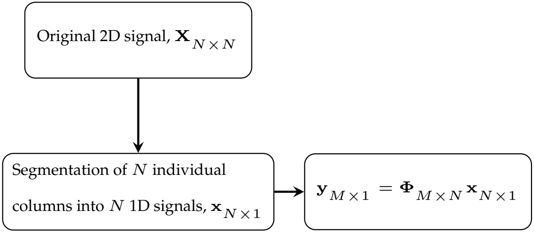
Original 2D signal, $\mathbf{X}_{N \times N}$

Column-wise Measurement

Original 2D signal, $\mathbf{X}_{N \times N}$

Segmentation of N individual columns into N 1D signals, $\mathbf{x}_{N \times 1}$

Column-wise Measurement

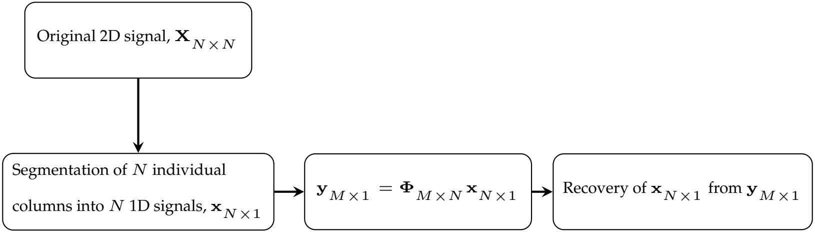


Original 2D signal, $\mathbf{X}_{N\times N}$

Segmentation of N individual columns into N 1D signals, $\mathbf{x}_{N\times 1}$

$\mathbf{y}_{M\times 1} = \mathbf{\Phi}_{M\times N}\mathbf{x}_{N\times 1}$

Column-wise Measurement



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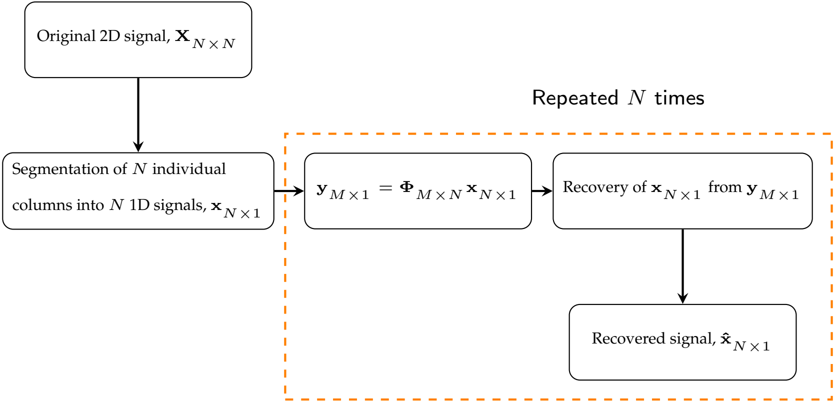
Segmentation of N individual columns into N 1D signals, $\mathbf{x}_{N \times 1}$

$\mathbf{y}_{M \times 1} = \mathbf{\Phi}_{M \times N} \mathbf{x}_{N \times 1}$

Recovery of $\mathbf{x}_{N \times 1}$ from $\mathbf{y}_{M \times 1}$

Column-wise Measurement



Original 2D signal, $\mathbf{X}_{N \times N}$

Segmentation of N individual columns into N 1D signals, $\mathbf{x}_{N \times 1}$

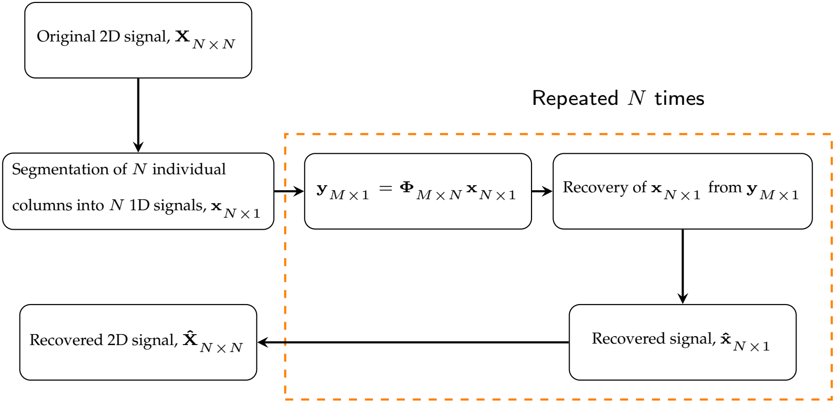
Repeated N times

$\mathbf{y}_{M \times 1} = \mathbf{\Phi}_{M \times N} \mathbf{x}_{N \times 1}$

Recovery of $\mathbf{x}_{N \times 1}$ from $\mathbf{y}_{M \times 1}$

Recovered signal, $\hat{\mathbf{x}}_{N \times 1}$

Column-wise Measurement



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Segmentation of N individual columns into N 1D signals, $\mathbf{x}_{N \times 1}$

Repeated N times

$\mathbf{y}_{M \times 1} = \mathbf{\Phi}_{M \times N} \mathbf{x}_{N \times 1}$

Recovery of $\mathbf{x}_{N \times 1}$ from $\mathbf{y}_{M \times 1}$

Recovered signal, $\hat{\mathbf{x}}_{N \times 1}$

Recovered 2D signal, $\hat{\mathbf{X}}_{N \times N}$

Column-wise Measurement and Segmentation of Image

Figure: Original Image [7]

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[7] Image freely available in Image Processing Toolbox ™ of MathWorks ®

Column-wise Measurement and Segmentation of Image

Column-wise Measurement and Segmentation of Image

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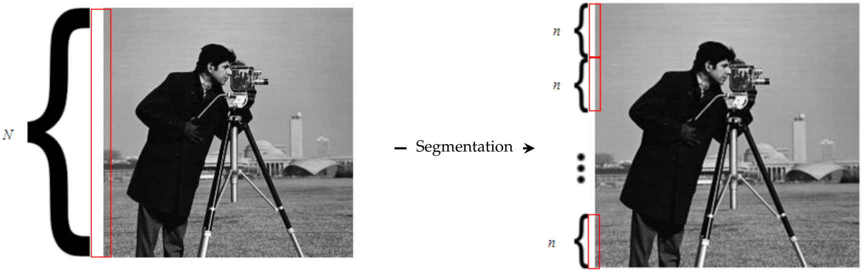

Modified Kronecker-based Recovery for 2-D Signals

(a) Reconstructed image using standard Kronecker-based technique

(b) Reconstructed image using modified Kronecker-based technique

Discussions

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Drawbacks

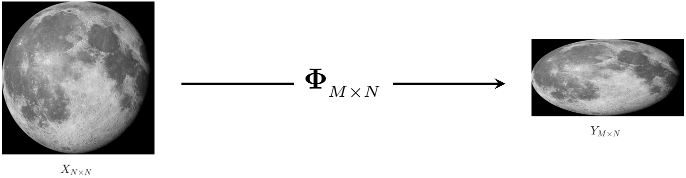
$X_{N \times N}$

Drawbacks

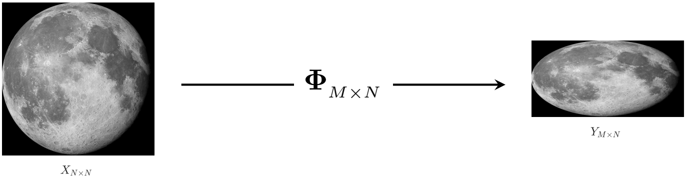
$X_{N \times N}$

$Y_{M \times N}$

Drawbacks



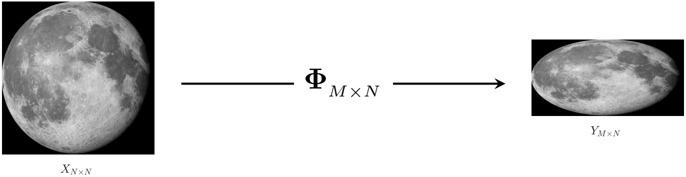
Drawbacks



Takeaways

- ✓ Aspect ratio of the compressed image not preserved.

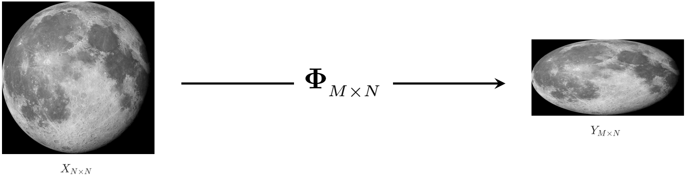
Drawbacks



Takeaways

- ✓ Aspect ratio of the compressed image not preserved.
- ✓ Compressed domain classification becomes difficult.

Drawbacks



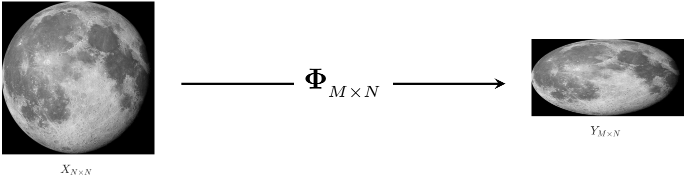
Takeaways

✓ Aspect ratio of the compressed image not preserved.

✓ Compressed domain classification becomes difficult.

✓ Images stored in the compressed domain required storage space for an $M \times N$ matrix.

Drawbacks



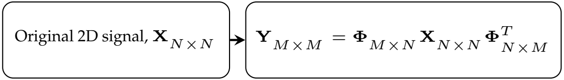
Takeaways

- ✓ Aspect ratio of the compressed image not preserved.
- ✓ Compressed domain classification becomes difficult.
- ✓ Images stored in the compressed domain required storage space for an $M \times N$ matrix.
- ✓ Measurement is required to be performed N times.

Row & column-wise Measurement

Original 2D signal, $\mathbf{X}_{N \times N}$

Row & column-wise Measurement



Original 2D signal, $\mathbf{X}_{N \times N}$

$$\mathbf{Y}_{M \times M} = \mathbf{\Phi}_{M \times N} \mathbf{X}_{N \times N} \mathbf{\Phi}^{T}_{N \times M}$$

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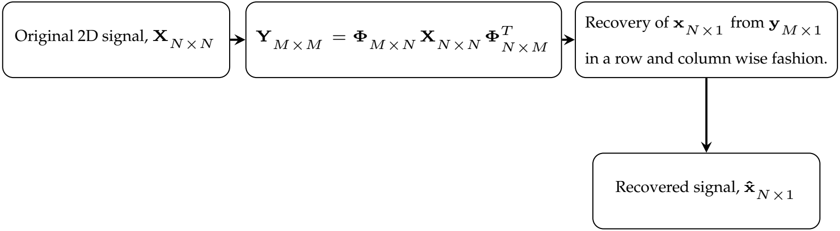
Recovery of $\mathbf{x}_{N \times 1}$ from $\mathbf{y}_{M \times 1}$ in a row and column wise fashion.

Row & column-wise Measurement



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Recovered signal, $\hat{\mathbf{x}}_{N \times 1}$

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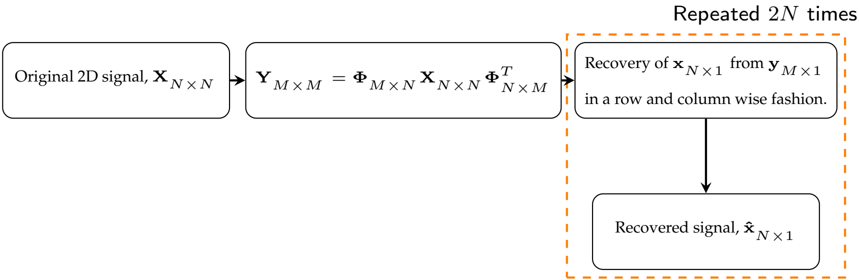

Original 2D signal, $\mathbf{X}_{N \times N}$

$\mathbf{Y}_{M \times M} = \mathbf{\Phi}_{M \times N} \mathbf{X}_{N \times N} \mathbf{\Phi}^{T}_{N \times M}$

Repeated $2N$ times

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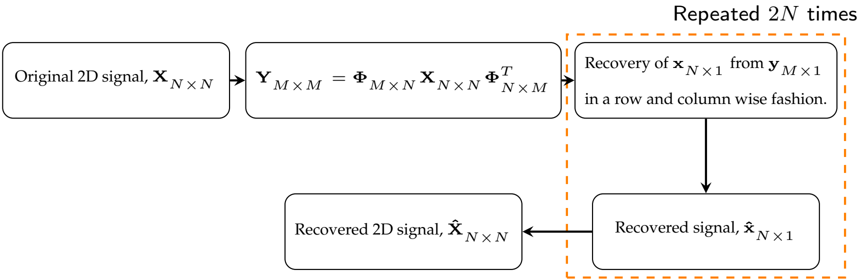
Recovered signal, $\hat{\mathbf{x}}_{N \times 1}$

Row & column-wise Measurement



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Recovered signal, $\hat{\mathbf{x}}_{N \times 1}$

Recovered 2D signal, $\hat{\mathbf{X}}_{N \times N}$

Advantages

$X_{N\times N}$

Advantages

$X_{N\times N}$

$Y_{M\times M}$

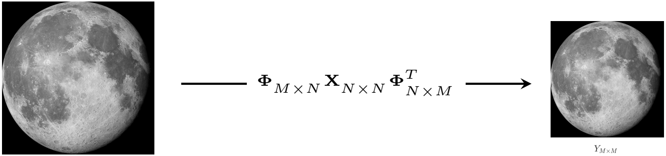
Advantages

$X_{N\times N}$

$$\xrightarrow{\mathbf{\Phi}_{M\times N}\,\mathbf{X}_{N\times N}\,\mathbf{\Phi}^{T}_{N\times M}}$$

$Y_{M\times M}$

Advantages



$$\boldsymbol{\Phi}_{M\times N}\,\mathbf{X}_{N\times N}\,\boldsymbol{\Phi}^{T}_{N\times M}$$

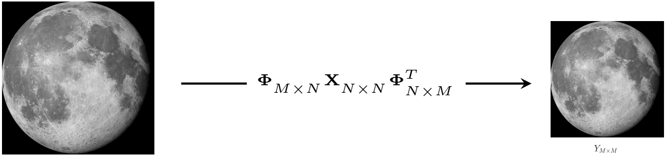
$X_{N\times N}$

$Y_{M\times M}$

Takeaways

✓ Aspect ratio of the compressed image is preserved.

Advantages



$X_{N\times N}$ $\xrightarrow{\mathbf{\Phi}_{M\times N}\mathbf{X}_{N\times N}\mathbf{\Phi}^{T}_{N\times M}}$ $Y_{M\times M}$

Takeaways

✓ Aspect ratio of the compressed image is preserved.

✓ Images stored in the compressed domain requires lesser storage space ($M \times M$ matrix), compared to the column-wise technique.

Row & Column-wise Recovery

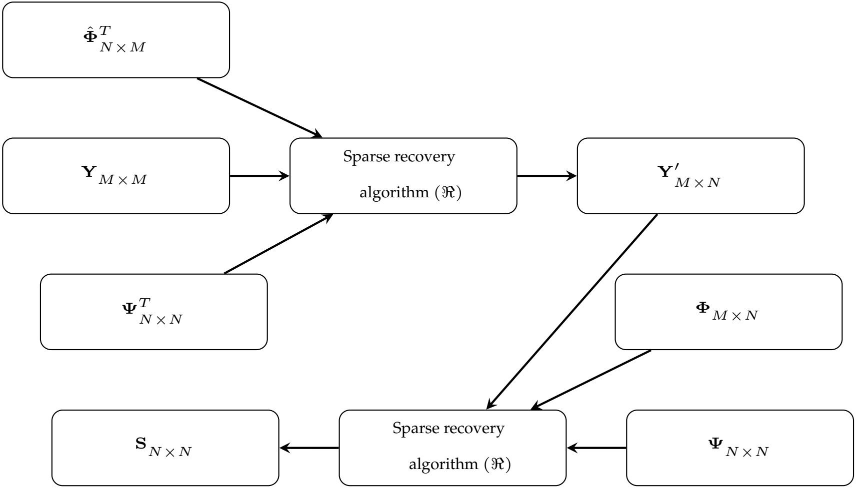
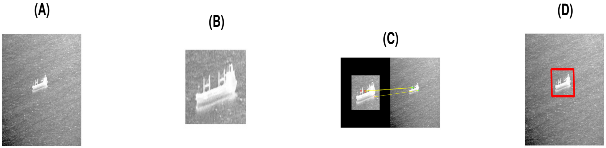


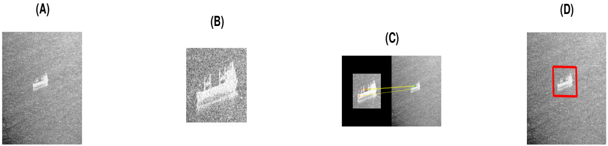
Figure: Block diagram representation of modified 2-D CS recovery.

Compressed Domain Template Matching

Infra-red Image (Without Noise)



Infra-red Image (With Noise[7])



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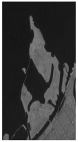
[7] White Gaussian noise of 0 mean and variance of 0.03 has been added.

Compressed Domain Template Matching

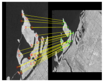
Medium Resolution Image (Without Noise)

(A)

(B)



(C)

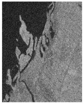


(D)

Compressed Domain Template Matching

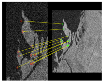
Medium Resolution Image (With Noise)

(A)

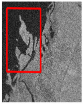


(B)

(C)



(D)



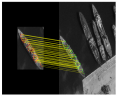
Compressed Domain Template Matching

High Resolution Image (Without Noise)

(A)

(B)

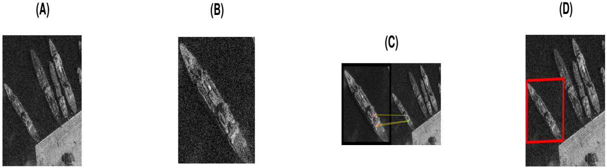
(C)



(D)

Compressed Domain Template Matching

High Resolution Image (With Noise)



Contributions

✓ Development of modified Kronecker-based recovery technique for deterministic CS for 1-D and 2-D signals.

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✓ Investigation of Kronecker-based CS recovery technique for ECG signals using various sparsifying dictionaries, measurement matrices and noise levels for various compression levels.

Publications

Journal Publications

1. **D. Mitra**, H. Zanddizari and S. Rajan, “Investigation of Kronecker-based Recovery of Compressed ECG Measurements," under revision after first submission to *IEEE Transactions on Instrumentation and Measurement*, 2019.
2. H. Sadreazami, **D. Mitra**, S. Rajan and M. Bolic, “Fall Detection in Compressed Domain using Machine Learning,” 2019. (Under Preparation).

Conference Publications

1. **D. Mitra**, S. Rajan and B. Balaji, “Deterministic compressive sensing approachfor compressed domain image analysis," *in 2019 IEEE Sensors Applications Symposium (SAS)*, France, March 2019.
2. **D. Mitra**, H. Zanddizari and S. Rajan, “Improvement of recovery in segmentation-based parallel compressive sensing,” *in 2018 IEEE International Symposium on Signal Processing and Information Technology (ISSPIT)*, pp. 501-506, Lousville, USA, Dec 2018.
3. **D. Mitra**, H. Zanddizari and S. Rajan, “Improvement of signal quality during recovery of compressively sensed ECG signals," *in 2018 IEEE International Symposium on Medical Measurements and Applications (MeMeA)*, pp. 1-5, Rome, Italy, June 2018.

Scope of Future Works

✓ Hardware-based implementation of Kronecker-based recovery and 1-D signal acquisition.

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✓ Possibility of Kronecker-based measurement for an extension to multi-dimensional signal processing (such as 3-D MRI).

Additional Results & Discussions

Compressibility Checking

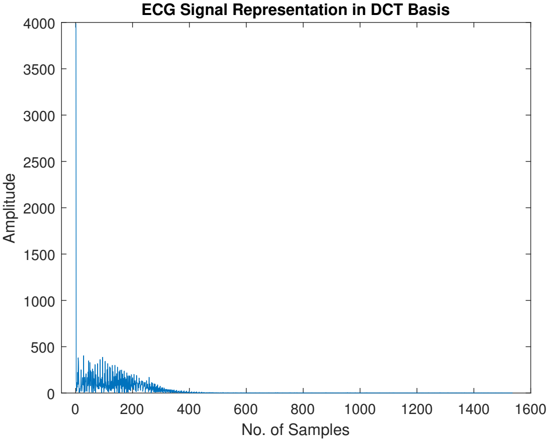

Figure: DCT domain representation of ECG signal

Quality Analysis for 2-D Signals

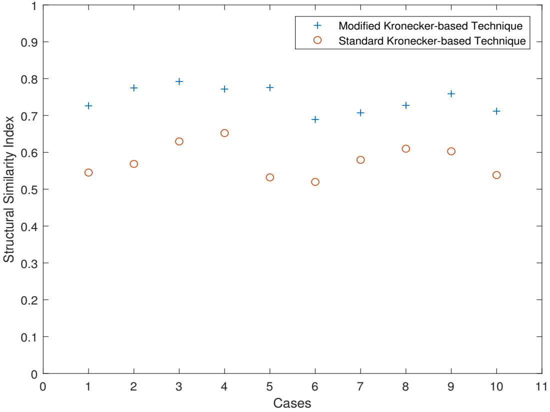

Figure: SSIM Analysis at CR = 93.75%

Quality Comparison for Segmentation

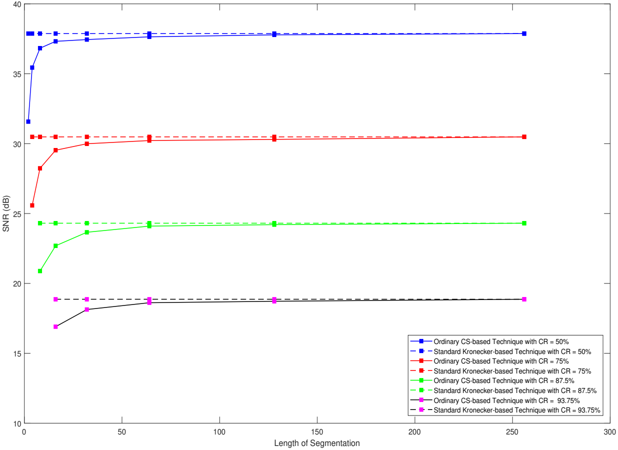

Figure: Reconstruction quality comparison between segmentation-based CS and ordinary CS methods.